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Internal waves produced by a convective layer

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Fluid impacts on the base of a stably stratified region of fluid cause internal-wave ripples whose spread is predominantly horizontal if the duration of the impacts is long compared with the natural period of the stratified fluid. The development of a single ripple in a slightly viscous fluid is calculated, first with a constant vertical gradient of potential density and then with a gradient varying linearly with height. The single-ripple results are used to find the intensity of the statistically steady wave motion generated by impacts which are randomly distributed in space and time. Above a critical height, dependent on the viscosity and stability of the fluid and on the time and length scales of the impacts, wave energy falls off as the $-5/3$ power of the height with a constant density gradient and as the $-25/6$ power with a linearly varying gradient. The predictions are compared with observations of temperature fluctuations in the stable region of an ice–water convection system and with observations of 'clear-air turbulence' over strato-cumulus cloud. Reasonable numerical agreement can be obtained with plausible values for the scales of the convective motion which provides the impacts.

1. Introduction

Laboratory observations of convection in a water–ice system show strong internal waves in the stably stratified region just above the region of active convection (Townsend 1964), and it is possible that the 'turbulence' reported by James (1959) and by Moore (1964) over strato-cumulus cloud is an internal wave motion generated by convection in the cloud. The large intensity and limited vertical extent of the waves was thought to be caused by the slow vertical spread of the ripples† excited by impacts of convective movements on the 'inversion', which stores energy, combined with viscous dissipation which destroys the ripples before they can spread far above the inversion. Theoretical reasons for the slow vertical spread were given briefly in the original paper, but it is not difficult to calculate both the development of a ripple excited by a single localized disturbance and the statistics of the stationary, random wave disturbance excited by a series of disturbances of random occurrence in horizontal position and time. As examples of convective layers adjoining regions of stable stratification are not uncommon in nature, the calculations may have more applications than the two quoted examples.

† 'Ripple' is used in the non-technical sense of the wave disturbance produced by a transient disturbance. No confusion with capillary waves should occur.

2. Periodic progressive waves

In a stably stratified inviscid fluid, internal waves with small vertical displacements given by

$$\zeta = \psi(z) \exp i(lx - \omega t)$$

can be propagated if $\psi(z)$ satisfies

$$d^2\psi/dz^2 + \{(\omega_0^2/\omega^2) - 1\} l^2 \psi = 0 \tag{2.1}$$

and the necessary boundary conditions (Lamb 1932). Oz is the vertical axis, and $\omega_0^2 = -(g/\rho)(d\rho/dz)$, where ρ is understood as potential density if the fluid is compressible. We consider the stratification defined by

$$\begin{aligned} \omega_0 &= \omega_1 \quad \text{for} \quad z > 0, \\ &= \omega_2 \quad \text{for} \quad z < 0, \end{aligned}$$

with density continuous across $z = 0$ and the fluid effectively unbounded in all directions. Two independent solutions of (2.1) are then

$$\left.\begin{aligned} \psi(z) &= \cos[\{(\omega_1^2/\omega^2) - 1\}^{\frac{1}{2}} lz] \quad \text{for} \quad z > 0, \\ &= \cos[\{(\omega_2^2/\omega^2) - 1\}^{\frac{1}{2}} lz] \quad \text{for} \quad z < 0, \end{aligned}\right\} \tag{2.2}$$

$$\left.\begin{aligned} \psi(z) &= \sin[\{(\omega_1^2/\omega^2) - 1\}^{\frac{1}{2}} lz] \quad \text{for} \quad z > 0, \\ &= \left(\frac{\omega_1^2 - \omega^2}{\omega_2^2 - \omega^2}\right)^{\frac{1}{2}} \sin[\{(\omega_2^2/\omega^2) - 1\}^{\frac{1}{2}} lz] \quad \text{for} \quad z < 0, \end{aligned}\right\} \tag{2.3}$$

and it is clear that all waves of frequency ω can be analysed into simple plane progressive waves with wave normals making angles of $\pm \tan^{-1}\{(\omega_0^2/\omega^2) - 1\}^{\frac{1}{2}}$ with the horizontal.

If the fluid is slightly viscous so that the viscous forces are all much less than the inertial and buoyancy forces, a travelling wave may have very nearly the inviscid form at any time but an amplitude decreasing exponentially with time. Necessary conditions are that the reduction of amplitude in a complete period should be small and that the ratio of wave energy to the local rate of energy loss by dissipation should be much the same everywhere. The second condition can be met either if $\omega_1 = \omega_2$ or if $\omega_2 = 0$; i.e. wave energy is either distributed uniformly or is negligible in most of the lower layer. A straightforward calculation of the rate of energy loss by viscous forces for the inviscid velocity distribution gives

$$\bar{\epsilon} = \tfrac{1}{2}\nu l^2 \omega_1^4 \omega^{-2} \quad \text{for} \quad z > 0, \tag{2.4}$$

where ν is the kinematic viscosity, and the average energy density is

$$\tfrac{1}{2}(\overline{\dot{\xi}^2} + \overline{\dot{\zeta}^2} + \omega_1^2 \overline{\zeta^2}) = \tfrac{1}{2}\omega_1^2, \tag{2.5}$$

where ξ is the horizontal displacement. It follows that the wave energy decays as $\exp(-\nu l^2 \omega_1^2 \omega^{-2} t)$ and that damped periodic waves exist of the form

$$\zeta = \psi(z|l, \omega) \exp\{i(lx - \omega t) - \tfrac{1}{2}\nu l^2 \omega_1^2 \omega^{-2} t\}. \tag{2.6}$$

The condition of weak viscous forces is met if

$$\omega^3/\nu\omega_1^2 l^2 \gg \tfrac{1}{2}. \tag{2.7}$$

3. Single ripples

The essential feature of a ripple is that the disturbance is highly concentrated in space at the initial instant $t = 0$. The effect can be obtained by superimposing waves of different frequencies and wave numbers to produce an initial disturbance of the required form. A symmetrical two-dimensional ripple that does not require an initial injection of fluid $\left(\text{i.e.} \int_{-\infty}^{\infty} \zeta\, dx = 0\right)$ is

$$\zeta_1 = \iint_{-\infty}^{\infty} \alpha^2 l^2 \exp\{-\tfrac{1}{2}(\alpha^2 l^2 + \tau^2\omega^2)\}\, \psi(z|l,\omega) \times \exp i(lx-\omega t) \exp(-\tfrac{1}{2}\nu l^2 \omega_1^2 \omega^{-2} t)\, dl\, d\omega, \tag{3.1}$$

where $$\psi(z|l,\omega) = \cos[\{(\omega_1^2/\omega^2)-1\}^{\frac{1}{2}} lz] \quad \text{for} \quad z > 0.$$

A circularly symmetric ripple could be obtained by substituting the Bessel function $H_0^{(1)}(lr)$ for $\exp i(lx)$, but, since

$$H_0^{(1)}(x) \sim \frac{\exp\{i(x-\frac{1}{4}\pi\}}{(\frac{1}{2}\pi x)^{\frac{1}{2}}}$$

for large x, the only significant difference is the appearance of a factor $(\frac{1}{2}\pi x)^{-\frac{1}{2}}$ in expressions for the wave amplitude. To show that ζ_1 satisfies our definition of a ripple, consider the disturbance

$$\zeta_0 = \iint_{-\infty}^{\infty} \psi(z|l,\omega) \exp\{-\tfrac{1}{2}(\alpha^2 l^2 + \tau^2\omega^2)\} \exp\{i(lx-\omega t) - \tfrac{1}{2}\nu l^2 \omega_1^2 \omega^{-2} t\}\, dl\, d\omega, \tag{3.2}$$

related by $$\zeta_1 = -\alpha^2 \frac{\partial^2 \zeta_0}{\partial x^2}. \tag{3.3}$$

Neglecting the viscous term,

$$\zeta_0 = \frac{2\pi}{\alpha\tau} \exp -\frac{1}{2}\left(\frac{x^2}{\alpha^2} + \frac{t^2}{\tau^2}\right) \tag{3.4}$$

on the plane $z = 0$, showing that α and τ are scales of length and time that measure the horizontal extent and duration of the initial disturbance. For $x = 0$,

$$\zeta_0 = \frac{\sqrt{(2\pi)}}{\alpha} \int_{-\infty}^{\infty} \exp[-\tfrac{1}{2}\{(\omega_1^2/\omega^2)-1\} z^2/\alpha^2] \exp(-i\omega t - \tfrac{1}{2}\tau^2\omega^2)\, d\omega,$$

which, for $\omega_1 \tau \gg 1$, reduces to

$$\zeta_0 = (2\pi/\alpha\tau) \exp(-\tfrac{1}{2}t^2/\tau^2) \exp\{-\tfrac{1}{2}(\omega_1^2\tau^2 - 1) z^2/\alpha^2\}, \tag{3.5}$$

showing the vertical extent of the initial disturbance to be of order $\alpha/\omega_1\tau$. A non-buoyant parcel of fluid arriving at the plane $z = 0$ with initial velocity α/τ would be stopped after rising a distance $\alpha/\omega_1\tau$ if it did not mix with the surrounding fluid. If the ripple is produced by upcurrents from a region of convection, τ should be interpreted as the duration of the up-current and α/τ as the velocity.

Approximate expressions for ζ_0 valid for large values of x/α can be obtained. After integration with respect to wave number, equation (3.2) becomes

$$\zeta_0 = \int_{-\infty}^{\infty} \frac{(\frac{1}{2}\pi)^{\frac{1}{2}}}{(\alpha^2+\nu\omega_1^2\omega^{-2}t)^{\frac{1}{2}}}\left\{\exp\left(-\frac{1}{2}\frac{[x-\{(\omega_1^2/\omega^2)-1\}^{\frac{1}{2}}z]^2}{\alpha^2+\nu\omega_1^2\omega^{-2}t}\right)\right.$$
$$\left.+\exp\left(-\frac{1}{2}\frac{[x+\{(\omega_1^2/\omega^2)-1\}^{\frac{1}{2}}z]^2}{\alpha^2+\nu\omega_1^2\omega^{-2}t}\right)\right\}\exp\left(-i\omega t-\tfrac{1}{2}\tau^2\omega^2\right)d\omega, \quad (3.6)$$

showing that contributions to ζ_0 from Fourier components with frequency ω are concentrated near the planes

$$z = \pm\{(\omega_1^2/\omega^2)-1\}^{\frac{1}{2}}x.$$

For large values of x/α, the approximation

$$x-\{(\omega_1^2/\omega^2)-1\}^{\frac{1}{2}}z = \frac{\omega_1^2 z}{\omega_r^2}\frac{\omega-\omega_r}{(\omega_1^2-\omega_r^2)^{\frac{1}{2}}},$$

where

$$\omega_r = \omega_1\{1+(x^2/z^2)\}^{-\frac{1}{2}},$$

can be used to complete the integration. Although there is no difficulty in writing down the result in full, it is convenient to impose the condition $\omega_1^2\tau^2 \gg 1$ and obtain a more manageable expression

$$\zeta_0 = \frac{\pi\omega_1 z}{x^2}\exp\left\{-\frac{1}{2}\left(\omega_1^2\tau^2\frac{z^2}{x^2}+\alpha^2\omega_1^2\frac{z^2t^2}{x^4}+\nu\omega_1^2\frac{t^3}{x^2}\right)\right\}\exp(-i\omega_r t). \quad (3.7)$$

It will be seen that ζ_0 oscillates rapidly in space and time as $\exp(-i\omega_r t)$ with amplitude varying more slowly and becoming very small if any of the conditions

$$z^2/x^2 \gg (\omega_1\tau)^{-2}, \quad \alpha^2\omega_1^2t^2 \gg x^4/z^2, \quad \omega_1^3t^3 \gg \omega_1 x^2/\nu \quad (3.8)$$

is satisfied. The more realistic disturbance ζ_1 can be found by differentiating only to the rapidly oscillating part of ζ_0, i.e.

$$\exp(-i\omega_r t) = \exp\{-i\omega_1 zt/(x^2+z^2)^{\frac{1}{2}}\} = \exp(-i\omega_1 zt/x) \quad \text{if} \quad \omega_1^2\tau^2 \gg 1.$$

Then

$$\zeta_1 = (\alpha^2\omega_1^2z^2t^2/x^4)\,\zeta_0. \quad (3.9)$$

With no viscous dissipation of wave energy, the characteristics are as follows. At a fixed position (x, z), oscillations of the appropriate frequency ω_r are observed which grow initially as the square of elapsed time, reach maximum intensity at $t = 2^{\frac{1}{2}}x^2/\alpha\omega_1 z$, and then die away rapidly. Along a line of constant z/x, the propagation velocity of the maximum is $(2)^{-\frac{1}{2}}\alpha\omega_1 z/x$, which is of order α/τ in the middle of the disturbed region. The disturbance is very small for z/x more than $(\omega_1\tau)^{-1}$ and is zero on the plane $z = 0$. With viscous dissipation, the wave intensity in the middle of the disturbed region begins to fall of rapidly for $x > r_c$, where

$$r_c = \alpha^3/(\nu\omega_1^2\tau^3). \quad (3.10)$$

4. Statistically steady motion

If a region of active convection exists below a strongly stable layer, the interface will be disturbed by up-currents which arrive with random spacing in time and in the horizontal plane, and a statistically steady pattern of waves will arise from the superposition of many ripples. With random arrival, the mean-square intensity of wave motion at any height is simply the product of the number of disturbances per unit area and time and the integral of the intensity of a single ripple over all horizontal area and all positive time at the particular height. Without viscosity, the intensity is independent of height but viscous forces destroy ripples after they have spread vertically a distance of order

$$z_c = \alpha^3/(\nu\omega_1^3\tau^4), \tag{4.1}$$

and the wave intensity falls off at greater heights. A simple argument leading to this form for z_c is that the vertical propagation velocity is about $\alpha/(\tau^2\omega_1)$ near the middle of the ripple and that, since the decay time is of order $\alpha^2/(\nu\omega_1^2\tau^2)$ for typical wave numbers and frequencies, z_c equals the product, which is $\alpha^3/(\nu\omega_1^3\tau^4)$.

To obtain the integral of $\zeta_1\zeta_1^*$ over a horizontal plane $z=$ const. and all positive time, two-dimensional ripples are suitable, leading to the same results as circular ripples. We take

$$\zeta_1\zeta_1^* = \pi^2\frac{\alpha^4\omega_1^6z^6t^4}{x^{12}}\exp-\left(\omega_1^2\tau^2\frac{z^2}{x^2}+\alpha^2\omega_1^2\frac{z^2t^2}{x^4}+\nu\omega_1^2\frac{t^3}{x^2}\right)$$

and find the integral for two special cases (i) $z \ll z_c$ and (ii) $z \gg z_c$. For the purposes of the integration, the effects of viscosity are negligible if $z \ll z_c$. Then

$$\zeta_1\zeta_1^* = \pi^2\frac{\alpha^4\omega_1^6z^6t^4}{x^{12}}\exp\left\{-\left(\omega_1^2\tau^2\frac{z^2}{x^2}+\alpha^2\omega_1^2\frac{z^2t^2}{x^4}\right)\right\}, \tag{4.2}$$

and

$$\int_{-\infty}^{\infty}\int_0^{\infty}\zeta_1\zeta_1^*\,dt\,dx = \frac{3\pi^3}{8}(\alpha\tau)^{-1}, \tag{4.3}$$

and is independent of height. If $z \gg z_c$, it follows from the conditions for small ripple amplitude (3.8) that $\alpha^2\omega_1^2z^2t^2/x^4 \ll 1$ over the ranges of x and t significant for the process of integration. Then

$$\zeta_1\zeta_1^* = \pi^2\frac{\alpha^6\omega_1^6z^6t^4}{x^{12}}\exp-\left(\omega_1^2\tau^2\frac{z^2}{x^2}+\nu\omega_1^2\frac{t^3}{x^2}\right), \tag{4.4}$$

and

$$\int_{-\infty}^{\infty}\int_0^{\infty}\zeta_1\zeta_1^*\,dt\,dx = \tfrac{1}{3}\pi^2(\tfrac{2}{3})!\,(\tfrac{17}{6})!\,(\alpha\tau)^{-1}(z/z_c)^{-\frac{5}{3}}, \tag{4.5}$$

indicating a fairly rapid decrease of mean-square displacement with height. The two limiting forms for the integrated intensity predict the same value for $z/z_c = 1{\cdot}144$. In the statistically steady wave system produced by random ripples of the assumed form, we should expect the mean square of the vertical displacement at $z = 1{\cdot}14z_c$ to be of order one half of $(\overline{\zeta^2})_0$, the value at $z=0$, and that, for large values of z/z_c, it would be

$$\overline{\zeta^2} = (\overline{\zeta^2})_0\left(\frac{z}{1{\cdot}14z_c}\right)^{-\frac{5}{3}}. \tag{4.6}$$

Other parameters of the motion may be derived in a similar way from the basic ripple equation (3.9). Two of particular interest are the intensities of the vertical and horizontal components of the particle velocity. In the basic ripple, the vertical velocity is very nearly

$$w_1 = -i\omega_r \zeta_1,$$

and so
$$\left.\begin{aligned}\int_{-\infty}^{\infty}\int_0^{\infty} w_1 w_1^* \,dt\,dx &= \frac{3\pi^3}{16}(\alpha\tau^3)^{-1} && \text{for} \quad z \ll z_c,\\ &= \tfrac{1}{3}\pi^2(\tfrac{2}{3})!\,(\tfrac{23}{6})!\,(\alpha\tau^3)^{-1}(z/z_c)^{-\frac{5}{3}} && \text{for} \quad z \gg z_c.\end{aligned}\right\} \tag{4.7}$$

For large values of z/z_c, the intensity is

$$\overline{w^2} = (\overline{w^2})_0 \left(\frac{z}{3{\cdot}88 z_c}\right)^{-\frac{5}{3}}. \tag{4.8}$$

To find the intensity of the horizontal component of velocity, use the equality of mean potential and mean kinetic energy in the linear wave system, i.e.

$$\tfrac{1}{2}\omega_1^2\overline{\zeta^2} = \tfrac{1}{2}(\overline{u^2}+\overline{w^2}),$$

where u is the projection of the velocity vector on the horizontal plane. For large values of $\omega_1^2\tau^2$, the intensity is

$$\left.\begin{aligned}\int_{-\infty}^{\infty}\int_0^{\infty} u_1 \overline{u_1^*} \,dt\,dx &= \frac{3\pi^3}{8}\frac{\omega_1^2}{\alpha\tau} && \text{for} \quad z \ll z_c,\\ &= \tfrac{1}{3}\pi^2(\tfrac{2}{3})!\,(\tfrac{17}{6})!\,(\omega_1^2/\alpha\tau)(z/z_c)^{-\frac{5}{3}} && \text{for} \quad z \gg z_c.\end{aligned}\right\} \tag{4.9}$$

Notice that the horizontal intensity is greater than the vertical intensity by a factor of order $\omega_1^2\tau^2$, more exactly $2\omega_1^2\tau^2$ if $z \ll z_c$, and $(6/23)\,\omega_1^2\tau^2$ if $z \gg z_c$, and is nearly equal to $\omega_1^2\overline{\zeta^2}$. In other words, the particle displacements and velocities are nearly confined to horizontal planes.

In some situations, active convection may be occurring below only a part of the horizontal plane $z = 0$, but waves may spread beyond the space above the convection. To estimate the extent of the lateral spreading, consider the wave motion produced by ripples with random distribution in time and origins uniformly distributed on the half-plane $x > 0$, $z = 0$. As before, the wave intensity is the product of the number of ripples initiated per unit time and area by the integral of the ripple intensity over all positive time and over the half-plane $x > r_0$, where r_0 is the perpendicular distance of the point of observation from the edge of the convective area. It is appropriate here to use the circular ripples analogous to the two-dimensional ones, which lead to

$$\zeta_1\zeta_1^* = \pi\frac{\alpha^4\omega_1^6 z^6 t^4}{\tau^{13}}\exp-\left(\omega_1^2\tau^2\frac{z^2}{r^2}+\alpha^2\omega_1^2\frac{z^2t^2}{r^4}+\nu\omega_1^2\frac{t^3}{r^2}\right) \tag{4.10}$$

for large values of r/α, where r is radial distance from the ripple origin. The constant is chosen to give the same value of

$$\int_{-\infty}^{\infty}\int_0^{t} \zeta_1\zeta_1^* \,dt\,dx$$

as the two-dimensional ripples. Integration over all positive time leads to

$$\left.\begin{aligned}\int_0^{\infty}\zeta_1\zeta_1^*\,dt &= \frac{3\pi^{\frac{3}{2}}}{8}\frac{\omega_1}{\alpha}\frac{z}{r^3}\exp(-\omega_1^2\tau^2 z^2/r^2) && \text{if} \quad r \ll r_c,\\ &= \tfrac{1}{3}\pi(\tfrac{2}{3})!\,\frac{\alpha^4\omega_1^{\frac{8}{3}}z^6}{\nu^{\frac{5}{3}}r^{\frac{29}{3}}}\exp(-\omega_1^2\tau^2 z^2/r^2) && \text{if} \quad r \gg r_c.\end{aligned}\right\} \tag{4.11}$$

For heights such that $\omega_1 \tau z/r_0 \ll 1$, the exponentials in equations (4.11) are always nearly equal to one and the space integrations lead to

$$\left.\begin{aligned}\int_{x>r_0}\int_0^\infty \zeta_1\zeta_1^*\,dt\,dA &= \tfrac{3}{4}\pi^{\frac{3}{2}}\frac{\omega_1}{\alpha}\frac{z}{r_0} && \text{for}\quad r_0 \ll r_c,\\ &= \{\pi^{\frac{3}{2}}(\tfrac{2}{3})!\,(\tfrac{10}{3})!/23(\tfrac{23}{6})!\}(1/\alpha\tau)(\omega_1\tau z/r_0)^6(r_0/r_c)^{-\frac{5}{3}} && \text{for}\quad r_0 \gg r_c,\end{aligned}\right\} \tag{4.12}$$

where $r_c = \alpha^3(\nu\omega_1^2\tau^3)^{-1}$. For heights large compared with $r_0(\omega_1\tau)^{-1}$, the space integrations are effectively over the whole half-plane $x > 0$, and the intensities are therefore one half of the intensities at the same height over the convective

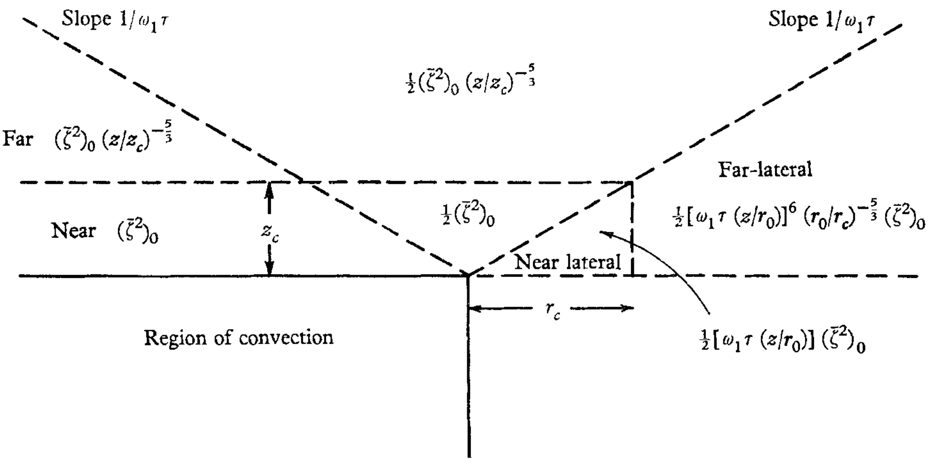

FIGURE 1. Distribution of wave intensities above and alongside a semi-infinite region of convection. (N.B. The wave intensities are correct only far from the boundaries of the several regions, which are shown as dotted lines. The numerical coefficients have been rounded up to simple fractions.)

area, as given in limiting conditions by equations (4.3) and (4.5). The distribution of intensity is shown diagrammatically in figure 1. Along the plane $r_0 = \omega_1\tau z$, the wave intensity is greater than elsewhere at the same value of r_0, but it falls off as $(r_0/r_c)^{-\frac{5}{3}}$ for large values of r_0. Broadly, appreciable wave displacements do not occur more than, say, $3r_c$ from the edge of the convective area or at a height of more than $3z_c$. Notice that $r_c = \omega_1\tau z_c$.

5. Ripples in a stratified fluid of varying density gradient

If the density gradient varies with height, progressive waves of simple harmonic form are not damped by viscous forces in a simple way, and the method of the previous section cannot be used. With not too rapid a variation of density gradient,† the difficulty can be overcome by building the ripple from wave packets of size small compared with $\omega_0/(d\omega_0/dz)$ but large compared with the mean wavelength in the packet. A two-dimensional wave packet with wave normals making an angle $-\theta$ with the horizontal has vertical displacements given by

$$\zeta = \iint \phi(k', m)\exp i\{(k+k')p + mq - (\omega+\omega')t\}\,dk'\,dm, \tag{5.1}$$

† The approximation is that of the W.K.B. method.

where $\phi(k', m)$ is non-zero only for values of $|k'|$ and $|m|$ much less than k, p is measured along the wave normal and q at right angles (figure 2). The frequency depends on k' and m, and, from (2.1),

$$\begin{aligned} \omega + \omega' &= \omega_0\{(k+k')\cos\theta + m\sin\theta\}/\{(k+k')^2+m^2\}^{\frac{1}{2}} \\ &= \omega_0\cos\theta + (\omega_0 m/k)\sin\theta \quad \text{if} \quad k'^2+m^2 \ll k^2. \end{aligned} \tag{5.2}$$

It follows from the dependence of frequency on wave number that the wave packet moves in the q-direction with group velocity $(\omega_0/k)\sin\theta$, while the waves move through the group in the p-direction with phase velocity $(\omega_0/k)\cos\theta$.

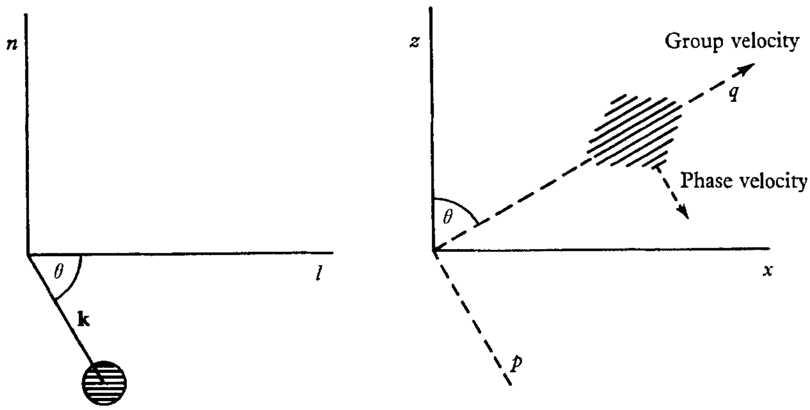

FIGURE 2. Composition of a wave packet, showing relation of group and phase velocities. Full, sloping lines are lines of constant phase, i.e. 'wave fronts'.

If the packet moves into a layer of different stability, i.e. with a different value of ω_0, the boundary conditions require that it conserves frequency and the horizontal component of wave number (see figure 3(*a*)). Then the vertical component of wave number is related to the constant horizontal component $l = k\cos\theta$ by

$$n = \{(\omega_0^2/\omega^2) - 1\}^{\frac{1}{2}}\, l = l\tan\theta, \tag{5.3}$$

and depends on the current position of the packet. An important consequence is that the vertical extent of the packet also changes and is proportional to the current value of $\cot\theta$. As a whole, the packet travels with group velocity $(\omega_0/k)\sin\theta = (\omega/l)\sin\theta$ in the direction $\frac{1}{2}\pi - \theta = \sin^{-1}(\omega/\omega_0)$, and its centre follows the trajectory defined by

$$\left.\begin{aligned} dz_t/dx_t &= \cot\theta = \omega/(\omega_0^2-\omega^2)^{\frac{1}{2}}, \\ dx_t/dt = (\omega_0/k)\sin^2\theta &= (\omega/l)\sin^2\theta = (\omega/l)\{1-(\omega^2/\omega_0^2)\}. \end{aligned}\right\} \tag{5.4}$$

Within the group, phase is propagated with the current phase velocity, which has a constant horizontal component ω/l and a vertical component $(\omega_0^2-\omega^2)^{\frac{1}{2}}/l$ (figure 3). The outcome is that a packet with the vertical displacements

$$\zeta = f(x, z)\exp i[lx + \{(\omega_1^2/\omega^2) - 1\}^{\frac{1}{2}} lz - \omega t]$$

near $t = 0$ ($f(x, z)$ is a slowly varying function of position with centroid at $x = z = 0$ and ω_1 is the value of ω_0 at $z = 0$) becomes at time t

$$\zeta = cf\left(x - x_t, \left\{\frac{\omega_0^2 - \omega^2}{\omega_1^2 - \omega^2}\right\}^{\frac{1}{2}} (z - z_t)\right) \exp i \left[lx + \int_0 \{(\omega_0^2/\omega^2) - 1\}^{\frac{1}{2}} l\, dz - \omega t\right], \quad (5.5)$$

where x_t, z_t are obtained by integrating equations (5.4), and c is a function of z_t. That is,

$$x_t = \int_0^{z_t} \frac{(\omega_0^2 - \omega^2)^{\frac{1}{2}}}{\omega}\, dz, \quad t = \frac{l}{\omega^2}\int_0^{z_t} \frac{\omega_0^2}{(\omega_0^2 - \omega^2)^{\frac{1}{2}}}\, dz. \quad (5.6)$$

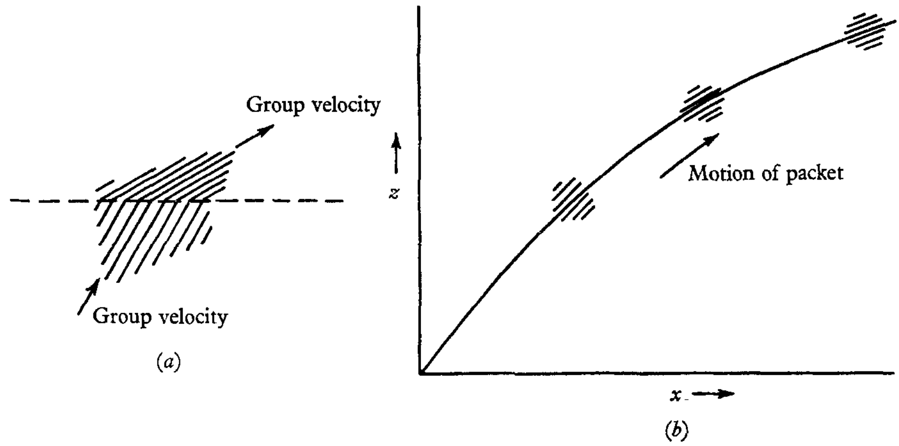

FIGURE 3. Propagation of a wave packet. (*a*) Refraction at the interface between two layers of different density gradients. (*b*) Motion of a wave packet along the trajectory. Full, sloping lines are lines of constant phase.

So far, the fluid has been supposed inviscid. If the fluid viscosity is small, the arguments used for a fluid of constant density gradient show that the logarithmic rate of decrease of the total energy of the packet is instantaneously $\nu l^2 \omega_0^2 \omega^{-2}$, i.e.

$$\frac{d}{dt} \log \iint \tfrac{1}{2}(\dot{\zeta}^2 + \dot{\xi}^2 + \omega_0^2 \zeta^2)\, dx\, dz = -\nu l^2 \omega_0^2 \omega^{-2}. \quad (5.7)$$

To find the energy at any time, the rate must be integrated along the trajectory and so

$$\iint \tfrac{1}{2}(\dot{\zeta}^2 + \dot{\xi}^2 + \omega_0^2 \zeta^2)\, dx\, dz \propto \exp\left\{-\int_0^{z_t} \frac{\nu l^3}{\omega^4} \frac{\omega_0^4}{(\omega_0^2 - \omega^2)^{\frac{1}{2}}}\, dz\right\}. \quad (5.8)$$

Remembering that mean potential energy equals mean kinetic energy in a wave motion and taking into account the change in vertical extent of the packet, the displacements at time t are given by

$$\zeta = f\left(x - x_t, \left\{\frac{\omega_0^2 - \omega^2}{\omega_1^2 - \omega^2}\right\}^{\frac{1}{2}} (z - z_t)\right) \exp\left\{-\frac{1}{2}\int_0^{z_t} \frac{\nu l^3}{\omega^4} \frac{\omega_0^4}{(\omega_0^2 - \omega^2)^{\frac{1}{2}}}\, dz\right\}$$

$$\times \frac{\omega_1(\omega_0^2 - \omega^2)^{\frac{1}{4}}}{\omega_0(\omega_1^2 - \omega^2)^{\frac{1}{4}}} \exp i \left[lx + \int_0^{z} \{(\omega_0^2/\omega^2) - 1\}^{\frac{1}{2}} l\, dz - \omega t\right]. \quad (5.9)$$

Ripples very similar to those considered in § 4 can be built by superimposing wave packets with a suitable distribution of horizontal wave number and frequency. The basic ripple, initially

$$\zeta_0 = \iint_{-\infty}^{\infty} \exp\{-\tfrac{1}{2}(\alpha^2 l^2 + \tau^2\omega^2)\} \cos[\{(\omega_1^2/\omega^2) - 1\}^{\frac{1}{2}} l z] \exp(ilx)\, dl\, d\omega,$$

becomes at time t

$$\zeta_0 = \iint_{-\infty}^{\infty} \frac{\omega_1}{\omega_0}\left(\frac{\omega_0^2-\omega^2}{\omega_1^2-\omega^2}\right)^{\frac{1}{2}} \exp\{-\tfrac{1}{2}(\alpha^2 l^2 + \tau^2\omega^2)\} \cos\left[\int_0^z \{(\omega_0^2/\omega^2) - 1\}^{\frac{1}{2}} l\, dz\right]$$
$$\times \exp\left\{-\tfrac{1}{2}\nu l^3 \omega^{-4} \int_0^{z_t} \frac{\omega_0^4}{(\omega_0^2-\omega^2)^{\frac{1}{2}}}\, dz\right\} \exp i(lx - \omega t)\, dl\, d\omega, \quad (5.10)$$

where x_t, z_t are the trajectory values for the appropriate frequency. Equation (4.2) is recovered if ω_0 is a constant.

Now consider the special case, $\omega_0^2 = \gamma z$, relevant to the ice–water convection system, and assume that $\omega^2 \ll \gamma z$ over the region considered. Then the trajectory equations (5.4) lead to

$$x_t = \tfrac{2}{3}\gamma^{\frac{1}{2}}\omega^{-1} z_t^{\frac{3}{2}} = \omega t/l, \quad (5.11)$$

and the viscous damping factor is

$$\exp-(\tfrac{1}{5}\nu l^3 \omega^{-4}\gamma^{\frac{3}{2}} z_t^{\frac{5}{2}}) = \exp-\{\tfrac{1}{5}(\tfrac{3}{2})^{\frac{5}{3}} \nu l^{\frac{4}{3}} \omega^{-\frac{2}{3}} \gamma^{\frac{2}{3}} t^{\frac{5}{3}}\}. \quad (5.12)$$

To simplify the integration over l, it is convenient to replace the viscous factor by

$$\exp-\tfrac{1}{2}\{\tfrac{3}{2}(\tfrac{3}{5})^{\frac{3}{2}} \nu^{\frac{3}{2}} \gamma \omega^{-1} t^{\frac{5}{2}} l^2\},$$

which coincides with the original at factor values of 1, $\exp(-\tfrac{1}{2})$ and 0. Putting

$$\alpha_e^2 = \alpha^2 + \tfrac{3}{2}(\tfrac{3}{5})^{\frac{3}{2}} \nu^{\frac{3}{2}} \gamma \omega^{-1} t^{\frac{5}{2}},$$

we find

$$\zeta_0 = \int_{-\infty}^{\infty} (\tfrac{1}{2}\pi)^{\frac{1}{2}} \left(\frac{\omega_1^2}{\gamma z}\right)^{\frac{1}{4}} \frac{1}{\alpha_e} \exp-\frac{1}{2}\left\{\frac{(\frac{2}{3}\gamma^{\frac{1}{2}} z^{\frac{3}{2}}/\omega) - x}{\alpha_e}\right\}^2 \exp(-\tfrac{1}{2}\tau^2\omega^2 - i\omega t)\, d\omega.$$

As before, contributions arise mostly from frequencies near

$$\omega_\tau = \tfrac{2}{3}\gamma^{\frac{1}{2}} z^{\frac{3}{2}}/x,$$

for which $\tfrac{2}{3}(\gamma^{\frac{1}{2}} z^{\frac{3}{2}}/\omega) - x = -\tfrac{2}{3}(\gamma^{\frac{1}{2}} z^{\frac{3}{2}}/\omega_\tau^2)(\omega - \omega_r)$

nearly. Then

$$\zeta_0 = \tfrac{2}{3}\pi\left(\frac{\omega_1^2}{\gamma z}\right)^{\frac{1}{4}} \frac{\gamma^{\frac{1}{2}} z^{\frac{3}{2}}}{x^2} \exp-\tfrac{1}{2}\left\{\tfrac{4}{9}\gamma\tau^2 \frac{z^3}{x^2} + \tfrac{4}{9}\gamma\alpha^2 \frac{z^3 t^2}{x^4} + (\tfrac{3}{5})^{\frac{3}{2}} \nu^{\frac{3}{2}} \gamma^{\frac{3}{2}} \frac{z^{\frac{3}{2}} t^{\frac{9}{2}}}{x^3}\right\} \exp(-i\omega_r t). \quad (5.13)$$

The more likely ripple is

$$\zeta_1 = -\alpha^2 \frac{\partial^2 \zeta_0}{\partial x^2} = \frac{4}{9} \frac{\alpha^2 \gamma z^3 t^2}{x^4} \zeta_0, \quad (5.14)$$

and $$\zeta_1 \zeta_1^* = \frac{64\pi^2}{729}\left(\frac{\omega_1^2}{\gamma z}\right)^{\frac{1}{2}} \frac{\alpha^4\gamma^3 z^9 t^4}{x^{12}} \exp-\left\{\tfrac{4}{9}\gamma\tau^2\frac{z^3}{x^2} + \tfrac{4}{9}\gamma\alpha^2\frac{z^3t^2}{x^4} + (\tfrac{3}{5})^{\frac{3}{2}}\nu^{\frac{3}{2}}\gamma^{\frac{3}{2}}\frac{z^{\frac{3}{2}}t^{\frac{9}{2}}}{x^3}\right\}. \quad (5.15)$$

To find the intensity of the statistically steady wave motion over a uniform distribution of ripple sources, we need the integral of $\zeta_1 \zeta_1^*$ over all positive time and all x. For $z \ll z_c$, where

$$z_c = \left(\frac{\alpha^6}{\nu^2 \gamma^3 \tau^8}\right)^{\frac{1}{5}}, \tag{5.16}$$

$$\int_{-\infty}^{\infty} \int_0^{\infty} \zeta_1 \zeta_1^* \, dt \, dx = \tfrac{3}{8}\pi^3 \left(\frac{\omega_1^2}{\gamma z}\right)^{\frac{1}{2}} \frac{1}{\alpha\tau}, \tag{5.17}$$

and, for $z \gg z_c$,

$$\int_{-\infty}^{\infty} \int_0^{\infty} \zeta_1 \zeta_1^* \, dt \, dx = \frac{\pi^2}{18} (\tfrac{2}{3})^{\frac{1}{3}} (\tfrac{5}{3})^{\frac{5}{3}} (\tfrac{1}{9})! \, (\tfrac{10}{3})! \left(\frac{\omega_1^2}{\gamma z}\right)^{\frac{1}{2}} \frac{1}{\alpha\tau} \left(\frac{z}{z_c}\right)^{-\frac{25}{6}}. \tag{5.18}$$

Notice that the 'near' disturbance diminishes as $z^{-\frac{1}{2}}$, an effect of the variation of density gradient. The effect of fluid viscosity is to reduce the mean square amplitude by a factor of $(1{\cdot}15z/z_c)^{-\frac{25}{6}}$ when z/z_c is large.

6. Application to real flows

The wave amplitudes have been calculated by assuming a statistically uniform distribution of disturbances at the bottom of the stable layer, all of the same form and specified by particular values of the characteristic length and time. The results obtained do not depend strongly on the assumed form except through these parameters and a wider class of disturbances could be treated by assuming various distributions of them. Unless initial disturbances with a higher degree of symmetry are used, most of the results are unchanged if suitably averaged, effective values of the parameters are substituted. Initial disturbances of lower symmetry, e.g. the basic disturbance ζ_0, involve injection of fluid from outside.

In practice, the disturbances are most likely to arise from the presence of convective motion below the stably stratified fluid, and it is necessary to consider the relation of the characteristic scales of length and time to the convective motion. If a weakly buoyant parcel of fluid enters the stable region moving with vertical velocity v_0, it may penetrate a distance of v_0/ω_1 before falling back if no mixing occurs. It has been shown that the initial disturbance (see equation (3.5)) extends into the stable fluid a distance of order $\alpha(\omega_1\tau)^{-1}$ if $\omega_1\tau$ is large, and so

$$v_0 \approx \alpha/\tau. \tag{6.1}$$

The implication is that the horizontal scale α is determined by the horizontal spreading of the parcel as it is retarded by the buoyancy forces. The time scale depends on the duration of the impact which, for a roughly spherical parcel, is about π/ω_1. Matching the simple harmonic time variation to the error function of equation (3.5), the effective value of τ is nearly ω_1^{-1} and $\omega_1\tau \approx 1$. Concentration of wave motion just above the bottom of the stable layer requires that $\omega_1^2\tau^2 \gg 1$, and the theory is applicable to convective-stable systems only if the convective motions are persistent columns rather than compact thermals. Then τ is the lifetime of a single column.

Let us examine now the relevance of the ripple theory of § 5 to the observed waves in the water–ice convection system (Townsend 1964). The arguments of the previous paragraph apply equally well to the stratification characteristic of the water–ice system, i.e. $\omega_0^2 = \gamma z$ for $z > 0$. It has been shown that the wave intensity decreases rapidly with height when z exceeds

$$z_c = \left(\frac{\alpha^6}{\nu^2\gamma^3\tau^8}\right)^{\frac{1}{5}}.$$

Study of the temperature records suggests that the critical height is nearly where the mean temperature is 6 °C, i.e. about 4 mm above the position of maximum density. Inserting the values

$$\gamma = 0{\cdot}36\,\text{cm}^{-1}\,\text{sec}^{-2}, \quad \nu = 0{\cdot}015\,\text{cm}^2\,\text{sec}^{-1},$$

we find

$$(\alpha^6/\tau^8)^{\frac{1}{5}} = 0{\cdot}04\,\text{cm}^{\frac{8}{5}}\,\text{sec}^{-\frac{8}{5}}.$$

From the temperature records, the dominant frequency is about $0{\cdot}1\,\text{rad}\,\text{sec}^{-1}$, corresponding to $\tau = 10\,\text{sec}$, and then $\alpha = 1{\cdot}5\,\text{cm}$ and $v_0 \approx 0{\cdot}15\,\text{cm}\,\text{sec}^{-1}$. These values of α and v_0 are completely plausible, and the condition that $\omega_0^2\tau^2 \gg 1$ near $z = z_c$ is well satisfied, i.e. $\omega_0^2\tau^2 = \gamma z_c \tau^2 = 14$.

The other real situation that may be represented by the ripple model is the motion in the air immediately above extensive layers of strato-cumulus cloud, which are strongly stable because of heat radiation from the cloud top. Flight observations described by James (1959) and by Moore (1964) show that 'clear-air turbulence' extends to about 100 m over the top, then becoming very small. Supposing the 'turbulence' to be a random wave motion excited by the convection in the cloud, the wave amplitude should decrease fairly rapidly beyond

$$z_c = \frac{\alpha^3}{K_m\omega_1^3\tau^4},$$

where K_m replaces ν as an effective coefficient of eddy viscosity. In terms of the velocity of the convective columns,

$$z_c = \frac{v_0^3}{K_m\omega_1^3\tau},$$

and, putting $\omega_1 = 6 \times 10^{-2}\,\text{sec}^{-1}$ (corresponding to the mean of the observed temperature gradients, about $100\,°\text{C}\,\text{km}^{-1}$) and assuming $v_0 = 1\,\text{m}\,\text{sec}^{-1}$ and $\tau = 100\,\text{sec}$, $z_c = 100\,\text{m}$ corresponds with an effective viscosity of

$$K_m = 5 \times 10^3\,\text{cm}^2\,\text{sec}^{-1},$$

well within the range of K_m suggested by James and Moore from other considerations. The value of $\omega_1^2\tau^2$ is 36, satisfactorily large. Moore reports temperature fluctuations just over the cloud tops with standard deviations of about 0·5 degC. In the gradient of $100\,°\text{C}\,\text{km}^{-1}$, these fluctuations would be caused by vertical displacements of 5 m, requiring an initial updraught of $0{\cdot}3\,\text{m}\,\text{sec}^{-1}$. Since the inversion is not perfectly sharp and since the temperature records appear to be rather intermittent, the argument underestimates the probable displacements and the result is in fair agreement with the suggested value of $v_0 = 1\,\text{m}\,\text{sec}^{-1}$.

The theory of § 4 suggests that internal waves can spread beyond the edges of a finite cloud confined below an inversion. The waves should be comparable in magnitude with those above the cloud within horizontal distances less than

$r_c = \omega_1 \tau z_c$. For the strato-cumulus layer, the distance is about 600 m. In less stable conditions the spread is greater.

The calculations have assumed an absence of wind shear in the stable layer, which is a good approximation only if the variation of wind in the disturbed layer is small compared with typical horizontal phase velocities, i.e. with $\alpha/\tau \approx v_0$. For the conditions above the strato-cumulus layer, the upper limit of wind shear is of order 10 m sec^{-1} km^{-1}.

7. Discussion

The good agreement of the theoretical predictions with the observations of the water–ice convection system implies that convection is carried out by rising columns of comparatively long lifetime whose vertical extent is large compared with their width and may be comparable with the total depth of the convective layer. Thermals, i.e. rising quasi-spherical parcels of buoyant fluid, could not satisfy the essential condition of large $\omega_1^2 \tau^2$, and it is likely that they can be the principal agents of convection only in a non-turbulent environment. In the troposphere, the normal state of stable stratification ensures that clear air is *relatively* non-turbulent and so thermals are common. Within a convective cloud of large horizontal extent, the environment of convective elements is likely to be highly turbulent, resembling the conditions found in free convection observed in the laboratory and in the earth's boundary layer (Townsend 1959; Priestley 1959). If the clear-air turbulence found over strato-cumulus cloud is interpreted as internal waves excited by convective movements in the cloud, the large value of $\omega_1^2 \tau^2$ implies that the convective processes exert sustained forces on the bottom of the stable layer. If the processes are mostly rising elements of warm air, they must be columns rather than thermals. If they are parcels of cooled air detaching themselves from the cloud top, the detachment must be a prolonged process, consistent with the formation of downward-moving columns but not excluding completely the possibility that downward-moving thermals may develop in the interior of the cloud.

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REFERENCES

James, D. G. 1959 *Quart. J. Roy. Met. Soc.* **85**, 120.

Lamb, H. 1932 *Hydrodynamics*, p. 378. Cambridge University Press.

Moore, J. G. 1964 *Met. Office Scientific Paper*, no. 19. H.M.S.O.

Priestley, C. H. B. 1959 *Turbulent transfer in the lower atmosphere.* University of Chicago Press.

Townsend, A. A. 1959 *J. Fluid Mech.* **5**, 209.

Townsend, A. A. 1964 *Quart. J. Roy. Met. Soc.* **90**, 248.